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SYSTEMS OF HOLOMORPHIC MULTIVALUED PROJECTIONS ON COMPLEX MANIFOLDS

by Kamil Drzyzga

Abstract. Let M be a submanifold of a connected Stein manifold X. We construct a global system of holomorphic multivalued projections $X \longrightarrow M$. In particular, for every locally bounded family $\mathcal{F} \subset \mathcal{O}(M)$ we get a continuous extension operator $\mathcal{F} \longrightarrow \mathcal{O}(X)$.

1. Introduction. Let M be a complex submanifold of a Stein manifold X. Using Bishop's ideas of multivalued projections we proved in [**4**] that for every domain $U \subset\subset X$ there exists a linear continuous extension operator $\mathcal{O}(M) \longrightarrow \mathcal{O}(U)$. Now, we will study the problem of existence of global holomorphic multivalued projections $X \longrightarrow M$ (see Definition 5.1 and Theorem 5.5). Note that in the paper [**2**] the author suggested that a holomorphic multivalued projections could exist. In particular, we prove that there is a continuous extension operator $\mathcal{F} \longrightarrow \mathcal{O}(X)$ for each locally bounded family $\mathcal{F} \subset \mathcal{O}(M)$ and moreover as an application we get a **linear** continuous extension operator $L^2(M)^1 \longrightarrow \mathcal{O}(X)$.

2. Auxiliary Results. Let M be a d-dimensional analytic subset of a connected Stein manifold X. In the sequel we denote by $\mathrm{Reg}M$ the set of regular points of M. For a compact $K \subset X$, its holomorphic hull (with respect to the space $\mathcal{O}(X)$ of all holomorphic functions on X) will be denoted by $\widehat{K}_{\mathcal{O}(X)}$. Put $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D} := \mathbb{D}(1)$.

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[1] $L^2_h(M) := \{f \in \mathcal{O}(M) : \int_M |f|^2 < \infty\}$.

Definition 2.1. Let $f \in \mathcal{O}(X, \mathbb{C}^k)$. We say that a set $P \subset P_0 := M \cap f^{-1}(\mathbb{D}^k)$ is an *analytic polyhedron* in M ($P \in \mathcal{P}(M, k, f)$) if $P \subset\subset M$ and P is the union of a family of connected components of P_0.

We say that an analytic polyhedron $P \in \mathcal{P}(M, k, f)$ is *special* if $d = k$.

Theorem 2.2 (cf. [**2**]). *Assume that $P \in \mathcal{P}(M, k, f)$ and $S \subset P$, $T \subset f^{-1}(\mathbb{D}^k)$ are compact. Then there exists a special analytic polyhedron $Q \in \mathcal{P}(M, d, g)$ such that $S \subset Q \subset P$ and $g(T) \subset \mathbb{D}^d$.*

Theorem 2.3 (cf. [**2**]). *Assume that X is Stein, $T \subset X$ is compact, and U is an open neighborhood of T such that $(U \setminus T) \cap \widehat{T}_{\mathcal{O}(X)} = \varnothing$. Let $\mathcal{A}$ stand for the closure of $\mathcal{O}(U)|_T$ in the space $\mathcal{C}(T)$ of all complex valued continuous functions on T. Then for every non-zero homomorphism $\xi : \mathcal{A} \longrightarrow \mathbb{C}$ there exists an $x_0 \in T$ such that $\xi(f) = f(x_0)$ for every $f \in \mathcal{A}$. Consequently (cf. [**1**], Chapter I, Section II, Corollary 10), if $w_1, \dots, w_m \in \mathcal{A}$ have no common zeros on T, then there exist $c_1, \dots, c_m \in \mathcal{A}$ such that $c_1 w_1 + \dots + c_m w_m = 1$.*

Definition 2.4 (cf. [**2**]). A continuous map $f : X \longrightarrow Y$, where X, Y are topological spaces, is called *almost proper* if each connected component of $f^{-1}(S)$ is compact for every compact subset S of Y.

Theorem 2.5 (cf. [**2**]). *Let Y be a 0-dimensional analytic subset of* $\operatorname{Reg}(M)$. *Then there exists an $f \in \mathcal{O}(X, \mathbb{C}^d)$ such that $f|_M$ is almost proper and the mapping f gives local coordinates on M at x for each $x \in Y$.*

Theorem 2.6 (cf. [**2**]). *Assume that M is pure d-dimensional and let $f \in \mathcal{O}(X, \mathbb{C}^d)$ be such that $f|_M$ is almost proper. Let $\{S_j\}_{j=1}^{\infty}$ be an increasing sequence of compact subsets of M, each of which has finitely many connected components and $\bigcup_{j=1}^{\infty} S_j = M$. Let $\alpha : \mathbb{N} \longrightarrow \mathbb{R}_{>0}$ such that*

$$S_j \subset F_j := M \cap f^{-1}(\overline{\mathbb{D}}^d(\alpha(j)))$$

for all $j \in \mathbb{N}$. Let H_j be the union of all those connected components of F_j which intersect S_j. Then H_j is compact. For each $j \in \mathbb{N}$ put

$$G_j := (H_{j+1} \cap F_j) \setminus H_j.$$

Let $\{g_j\}_{j=1}^{\infty} \subset \mathcal{O}(M)$ and $\{\varepsilon_j\}_{j=1}^{\infty} \subset \mathbb{R}_{>0}$. Then there exists an $s \in \mathcal{O}(M)$ such that

$$|s(x) - g_j(x)| < \varepsilon_j, \quad x \in G_j, \quad j \in \mathbb{N}.$$

Moreover, given a countable set $A \subset M$, the function s can be chosen to have different values modulo $2\pi i$, i.e. $\mathrm{e}^{s(x)} \neq \mathrm{e}^{s(y)}$ for $x, y \in M$ and $x \neq y$.

Remark 2.7. Observe that:

(a) $H_j \subset H_{j+1}$ for $j \in \mathbb{N}$;
(b) $\bigcup_{j=1}^{\infty} H_j = M$.

3. Symmetric products. The aim of this section is to present some properties of the symmetric products. Details can be found in [**7**], Appendix V.

Let X be a Hausdorff topological space. We define an equivalence relation on X^k by $(x_1, \dots, x_k) \sim (y_1, \dots, y_k) :\Longleftrightarrow (y_1, \dots, y_k)$ is a reordering of $(x_1, \dots, x_k)$. $\overleftrightarrow{X^k} := X^k/\sim$ is called the k-symmetric product of X. In the case $k = 1$, we get $\overleftrightarrow{X^1} = X$. Now, we define the projection $\pi : X^k \longrightarrow \overleftrightarrow{X^k}$, $\pi(x) := [x]$. We put $[\![x_1, \dots, x_k]\!] := [(x_1, \dots, x_k)]$, $\{[\![x_1, \dots, x_k]\!]\} := \{x_1, \dots, x_k\}$. Moreover, we put

$$[\![x_1 : \mu_1, \dots, x_\ell : \mu_\ell]\!] := [\![\overbrace{x_1, \dots, x_1}^{\mu_1\text{-times}}, \dots, \overbrace{x_\ell, \dots, x_\ell}^{\mu_\ell\text{-times}}]\!],$$

provided that $x_j \neq x_t$ for $j \neq t$, $\mu_1, \dots, \mu_\ell \in \mathbb{N}$, $\mu_1 + \dots + \mu_\ell = k$. We define

$$[\![A_1, \dots, A_k]\!] := \Big\{[\![x_1, \dots, x_k]\!] : x_i \in A_i, \quad i = 1, \dots, k\Big\}.$$

The topology on $\overleftrightarrow{X^k}$ is defined by the basis

$$[\![U_1, \dots, U_m]\!], \quad U_i \text{ is open in } X, \quad i = 1, \dots, k.$$

Observe that π is continuous, open, and $\overleftrightarrow{X^k}$ is Hausdorff.

Definition 3.1. Let Y be Hausdorff topological space and let $F : X \longrightarrow \overleftrightarrow{Y^n}$ be continuous. Then we put

$$X_F^{(k)} := \{x \in X : \#\{F(x)\} = k\},$$

$$\chi_F := \max\{k : X_F^{(k)} \neq \varnothing\}, \quad X_F := X_F^{(\chi_F)}.$$

Note that X_F is open.

Proposition 3.2. *Let F be as above. Suppose that*

$$a \in X_F, \quad F(a) = [\![b_1 : \mu_1, \dots, b_\ell : \mu_\ell]\!], \quad \mu_1 + \dots + \mu_\ell = k \quad \ell := \chi_F.$$

Then there is a neighborhood $U \subset X_F$ of a and there are uniquely defined continuous functions $f_i : U \longrightarrow Y$, $i = 1, \dots, \ell$, such that

$$F(x) = [\![f_1(x) : \mu_1, \dots, f_\ell(x) : \mu_\ell]\!], \quad x \in U.$$

In the above situation, we will write $F = \mu_1 f_1 \oplus \dots \oplus \mu_\ell f_\ell$ on U.

Proposition 3.3. *Let $F : X^k \longrightarrow Y$ be continuous. Then F is symmetric if and only if there exists a continuous function $\overleftrightarrow{F} : \overleftrightarrow{X^k} \longrightarrow Y$ such that $F = \overleftrightarrow{F} \circ \pi$.*

4. Holomorphic multivalued functions and system of multivalued projections. All propositions below and their proofs are taken from [**4**]. We recall only those facts which will be used in this paper.

Definition 4.1. Let M, N be complex manifolds with M connected. We say a continuous mapping $F\colon M \longrightarrow \overleftrightarrow{N^n}$ is *holomorphic on* M ($F \in \mathcal{O}(M, \overleftrightarrow{N^n})$) if:

- $M \setminus M_F$ is thin, i.e. every point $x_0 \in M \setminus M_F$ has open connected neighborhood $V \subset M$ and a function $\varphi \in \mathcal{O}(V)$, $\varphi \not\equiv 0$, such that $(M \setminus M_F) \cap V \subset \varphi^{-1}(0)$,
- for every $a \in M_F$, if $F = \mu_1 f_1 \oplus \cdots \oplus \mu_\ell f_\ell$ on V as in Proposition 3.2, then $f_1, \ldots, f_\ell \in \mathcal{O}(V)$.

If M is disconnected, then we say that F is *holomorphic on* M if $F|_C \in \mathcal{O}(C, \overleftrightarrow{N^n})$ for any connected component $C \subset M$.

Proposition 4.2. *Let M, N, K be complex manifolds and let $f \in \mathcal{O}(M, N)$, $g \in \mathcal{O}(N, \overleftrightarrow{K^n})$. Assume that $f(M) \cap N_g \neq \varnothing$ and M is connected. Then $g \circ f \in \mathcal{O}(M_{g\circ f}, \overleftrightarrow{K^n})$.*

Proposition 4.3. *Let $f \in \mathcal{O}(M, \overleftrightarrow{N^n})$ and $g \in \mathcal{O}(N^n, K)$ be symmetric. Then $\overleftrightarrow{g} \circ f \in \mathcal{O}(M, K)$.*

Theorem 4.4 (cf. [**2**]; see also [**6**], Chapter 7). *Assume that $P \in \mathcal{P}(M, d, f)$ is special. Then there exist a $k \in \mathbb{N}$ and a holomorphic mapping $\omega : \mathbb{D}^d \longrightarrow \overleftrightarrow{P^k}$ such that:*

- *$f^{-1}(z) \cap P = \{\omega(z)\}$, $z \in \mathbb{D}^d$,*
- *$\#\{\omega(z)\} = k$ for $z \in \mathbb{D}^d \setminus \Sigma'$, where Σ' is a proper analytic set.*

The number k in the above theorem is called the *multiplicity of f on P*.

Definition 4.5. Let M be an analytic submanifold of a manifold X. Let $U \subset X$ be a domain such that $U \cap M \neq \varnothing$. We say a holomorphic function

$$\Delta : U \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^n}$$

is a *holomorphic multivalued projection* $U \longrightarrow M$ if for any $x \in U \cap M$ such that $\Delta(x) = [\![(x_1, z_1), \ldots, (x_n, z_n)]\!]$ we have $x_{j_0} = x$ for some $j_0 \in \{1, \ldots, n\}$ and $z_j = 0$ for any $j \in \{1, 2, \ldots, n\} \setminus \{j_0\}$.

Let $\mathfrak{P}$ denote the set of all holomorphic multivalued projections $U \longrightarrow M$. Then we define the map

$$\Xi : (U \cap M) \times \mathfrak{P} \longrightarrow \mathbb{C}, \quad \Xi(x, \Delta) := z_{j_0}.$$

Observe that Ξ is well defined.

Definition 4.6. We say $\Pi = (\Delta_s)_{s=1}^k$ is a *system of holomorphic multivalued projections* $U \longrightarrow M$ if $\Delta_s : U \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^{k_s}}$, $s = 1, \dots, k$, are holomorphic multivalued projections and $\sum_{s=1}^k \Xi(x, \Delta_s) = 1$ for any $x \in U \cap M$.

Theorem 4.7. *Assume that there exists a system Π of holomorphic multivalued projections on U. Then there exists a linear continuous operator*

$$L_\Pi : \mathcal{O}(M) \longrightarrow \mathcal{O}(U)$$

such that $L_\Pi(u)(x) = u(x)$ for $x \in U \cap M$.

Theorem 4.8. *Let M be an analytic submanifold of a Stein manifold X. Let U be a relatively compact domain of X such that $U \cap M \neq \varnothing$. Then there exists a system of multivalued holomorphic projections $U \longrightarrow M$.*

Theorems 4.7 and 4.8 immediately imply the following result.

Theorem 4.9. *Let M be an analytic submanifold of a Stein manifold X. Let U be a relatively compact domain of X such that $U \cap M \neq \varnothing$. Then there exists a linear continuous extension operator $L : \mathcal{O}(M) \longrightarrow \mathcal{O}(U)$.*

Proposition 4.10. *Let ω, f, X,P be as above. Additionally assume that $f(U) \subset \mathbb{D}^d$, where $U \subset X$ is a domain and $U \cap P \neq \varnothing$. Then $\omega \circ f|_U \in \mathcal{O}(U, \overleftrightarrow{P^k})$.*

Proposition 4.11. *Let ω, f, X,P be as above. Then $\omega \circ f|_P \in \mathcal{O}(P, \overleftrightarrow{P^k})$.*

5. Global system of holomorphic multivalued projections. Let X be a connected complex manifold and M be a complex submanifold.

Definition 5.1. A sequence $\Pi = (\Delta_{s,j})_{(s,j) \in \{1,\dots,r\} \times \mathbb{N}}$ is called a *global system of holomorphic multivalued projections* $X \longrightarrow M$ if for each $j \in \mathbb{N}$ the mapping $\Delta_{s,j} : U_j \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^{k_{s,j}}}$ ($k_{s,j} \in \mathbb{N}$) is a holomorphic multivalued projection (in the sense of Definition 4.5), $s = 1, \dots r$, having the following properties

(a) $U_j \subset X$ is a domain with $U_j \cap M \neq \varnothing$, $U_j \subset U_{j+1}$, $\bigcup_{j \in \mathbb{N}} U_j = X$;
(b) $\lim_{n \to \infty} \sum_{s=1}^r \Xi(x, \Delta_{s,n}) = 1$, $x \in M$.

Remark 5.2. Let $\Pi = (\Delta_{s,j})_{(s,j) \in \{1,\dots,r\} \times \mathbb{N}}$ be as above.

(a) For each $j \in \mathbb{N}$ we get a linear continuous operator (cf. the proof of Theorem 4.7 in [**4**].)

$$L_{\Pi,j} : \mathcal{O}(M) \longrightarrow \mathcal{O}(U_j), \quad L_{\Pi,j} := \sum_{s=1}^r \overleftrightarrow{u_{s,j}} \circ \Delta_{s,j}, \text{ where}$$

$$\overleftrightarrow{u_{s,j}} : \overleftrightarrow{(M \times \mathbb{C})^{k_{s,j}}} \longrightarrow \mathbb{C}, \quad \overleftrightarrow{u_{s,j}}([\![(\xi_1, \lambda_1), \dots, (\xi_{k_{s,j}}, \lambda_{k_{s,j}})]\!]) = \sum_{m=1}^{k_{s,j}} u(\xi_m)\lambda_m.$$

(b) Using Definition 5.1(b), for $u \in \mathcal{O}(M)$ and $x \in M$ we get

$$\lim_{j\to\infty} L_{\Pi,j}(u)(x) = \lim_{j\to\infty} \sum_{s=1}^{r} \overleftrightarrow{u_{s,j}} \circ \Delta_{s,j}(x) = \lim_{j\to\infty} \sum_{s=1}^{r} u(x)\Xi(x, \Delta_{s,j}) = u(x).$$

Let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(M)$.

Definition 5.3. We say a global system of holomorphic multivalued projections $\Pi = (\Delta_{s,j})_{(s,j)\in\{1,\dots,r\}\times\mathbb{N}}$ is an $\mathcal{F}$-extension if for each $u \in \mathcal{F}$ the sequence $(L_{\Pi,j}(u))_{j=1}^{\infty}$ converges locally uniformly in X.

Set $L_\Pi(u) := \lim_{j\to\infty} L_{\Pi,j}(u)$, $u \in \mathcal{F}$.

Remark 5.4. Let $\Pi = (\Delta_{s,j})_{(s,j)\in\{1,\dots,r\}\times\mathbb{N}}$ be an $\mathcal{F}$-extension.
(a) By Remark 5.2(b), $L_\Pi : \mathcal{F} \longrightarrow \mathcal{O}(X)$ is a extension operator.
(b) If $u, v \in \mathcal{F}$ and $u + v \in \mathcal{F}$, then $L_\Pi(u + v) = L_\Pi(u) + L_\Pi(v)$.
(c) If $u \in \mathcal{F}$, $\alpha \in \mathbb{C}$ and $\alpha u \in \mathcal{F}$, then $L_\Pi(\alpha u) = \alpha L_\Pi(u)$.
(d) If $\mathcal{F}$ is a vector space, then L_Π is linear.
(e) If $u_1, \dots, u_m \in \mathcal{F}$ are linearly independent (in $\mathcal{O}(M)$), then the formula

$$L_\Pi(\alpha_1 u_1 + \cdots + \alpha_m u_m) := \alpha_1 L_\Pi(u_1) + \cdots + \alpha_m L_\Pi(u_m), \quad \alpha_1, \dots, \alpha_m \in \mathbb{C},$$

extends the operator L_Π to the vector space $\operatorname{span}\{u_1, \dots, u_m\}$.

The main result of the paper is the following theorem.

Theorem 5.5. *Let X be a Stein manifold and $\mathcal{F} \subset \mathcal{O}(M)$ be locally bounded (i.e. $\sup_{u\in\mathcal{F}} \|u\|_K < +\infty$[2] for every compact set $K \subset M$, e.g. $\mathcal{F}$ is finite). Then there exists an $\mathcal{F}$-extension $\Pi = (\Delta_{s,j})_{(s,j)\in\{1,\dots,d\}\times\mathbb{N}}$ with $d := \dim M$. Consequently, there exists a continuous extension operator $L_\Pi : \mathcal{F} \longrightarrow \mathcal{O}(X)$.*

Corollary 5.6. *Let X be a Stein manifold and $\mathcal{V}$ be a finitely dimensional vector subspace of $\mathcal{O}(M)$. Then there exists a linear continuous extension operator $L : \mathcal{V} \longrightarrow \mathcal{O}(X)$.*

Proposition 5.7. *Assume that $\mathcal{H} \subset \mathcal{O}(M)$ is a Hilbert space such that the unit ball $B := \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq 1\}$ is locally uniformly bounded and the convergence in the sense of $\mathcal{H}$ implies the locally uniform convergence in M. Then there exists a linear continuous extension operator $L : \mathcal{H} \longrightarrow \mathcal{O}(X)$. In particular, there exists a linear continuous extension operator $L : L^2_h(M) \longrightarrow \mathcal{O}(X)$.*

Proof. We put $\mathcal{F} := B$. By Theorem 5.5 there exists a continuous extension operator $\widetilde{L} : \mathcal{F} \longrightarrow \mathcal{O}(X)$. Moreover, since $\operatorname{span}(\mathcal{F}) = \mathcal{H}$, we conclude that there exists a linear continuous extension operator $L : \mathcal{H} \longrightarrow \mathcal{O}(X)$.

[2] $\|f\|_K := \sup_K |f|$.

Indeed, suppose that $(f_j)_{j=1}^{\infty} \subset \mathcal{F}$ is an orthonormal basis of $\mathcal{H}$. Set $\widetilde{f_j} := \widetilde{L}(f_j)$. Let $f \in \mathcal{H}$ be such that $f = \sum_{j=1}^{\infty} c_j f_j$. Put

- $L(f) := \sum_{j=1}^{\infty} c_j \widetilde{f_j} = C_f \widetilde{L}(f/C_f)$,
- $s_N := \sum_{j=1}^{N} c_j f_j$,

where $C_f := \|f\|_{\mathcal{H}}$. Since $(f_j)_{j=1}^{\infty}$ is orthonormal, hence

- $f_j, \frac{c_j}{C_f} f_j \in \mathcal{F}$,
- $\frac{c_j}{C_f} f_j + \frac{c_k}{C_f} f_k \in \mathcal{F}$ for $j, k \in \mathbb{N}$, $j \neq k$.

Therefore, $C_f \widetilde{L}(s_N/C_f) = \sum_{j=1}^{N} c_j \widetilde{f_j}$. As $s_N/C_f \longrightarrow f/C_f$ locally uniformly and $s_N/C_f \in \mathcal{F}$, we get $L(f) = C_f \widetilde{L}(f/C_f)$. By assumption on topologies, L is continuous.

Now, assume that $\mathcal{H} = L_h^2(M)$. It is known that for any compact set $K \subset M$ there are $C_K > 0$ and open neighborhood $K \subset \Omega \subset\subset M$ such that $\|f\|_K \leq C_K \|f\|_{L^2(\Omega, dV)}$. It follows that B is locally uniformly bounded. $\square$

Corollary 5.8. *Let $X \in \{\mathbb{D}^n, \mathbb{B}_n\}$. There exists a linear continuous extension operator $L : L_h^2(\mathbb{D})^3 \longrightarrow \mathcal{O}(X)$.*

Proof of Theorem 5.5. Let Y be an arbitrary 0-dimensional analytic subset of M. By Theorem 2.5 there exists a mapping $f \in \mathcal{O}(X, \mathbb{C}^d)$ such that $f|_M$ is almost proper and for each $x \in Y$ the mapping f gives local coordinates on M at x.

Let $S_k, \alpha(k), F_k, H_k$ and G_k be as in Theorem 2.6. Observe that $Q_k := \operatorname{int} H_k = H_k \cap f^{-1}(\mathbb{D}^d(\alpha(k))$ is a special analytic polyhedron. Let λ_k-denote the multiplicity of f in Q_k, defined via Theorem 4.4 with $\omega_k : \mathbb{D}^d(\alpha(k)) \longrightarrow \overleftrightarrow{Q_k^{\lambda_k}}$. Set $\omega_k(f(x)) = [x_1^k, ..., x_{\lambda_k}^k]$ (counted with multiplicities), $x \in f^{-1}(\mathbb{D}^d(\alpha(k)))$.

Observe that for arbitrary $x \in X$, the set $M \cap f^{-1}(f(x))$ is discrete. Let $(x_\nu)_{\nu=1}^{\infty} = M \cap f^{-1}(f(x))$ (points are counted with multiplicities). We assume that $x_1 = x$ for $x \in M$. Let

$$\Xi_k(x) := \{j \in \mathbb{N} : x_j \in H_k\}.$$

Observe that for each $k \in \mathbb{N}$ and $x \in f^{-1}(Q_k)$ the set $\Xi_k(x)$ is finite and $\{x_j : j \in \Xi_k(x)\} = \{x_1^k, \dots, x_{\lambda_k}^k\}$.

Put $g_k := \lambda_{k+1} + k^2 + 1$, $k \in \mathbb{N}$. By Theorem 2.6 there exists an $f_{d+1} \in \mathcal{O}(X)$ such that $|f_{d+1} - g_k| < 1$ on G_k, $k \in \mathbb{N}$, and the function $w := \mathrm{e}^{-f_{d+1}}$ separates points in $M \cap f^{-1}(f(x))$ for all $x \in Y$.

[3] $\mathbb{D} \simeq \{(z, \mathbf{0}) \in \mathbb{C}^n : z \in \mathbb{D}\}$.

Lemma 5.9. *Let $\mathcal{F} \subset \mathcal{O}(M)$ be locally bounded. Then there exists a function $f^*_{d+1} \in \mathcal{O}(X)$ such that if $h := \mathrm{e}^{-f^*_{d+1}}$ and*

$$\widetilde{\varphi}_k(x) := \sum_{j \in \Xi_k(x)} \varphi(x_j)h(x_j) \prod_{\substack{\mu \in \Xi_k(x) \\ \mu \neq j}} \Big(1 - \frac{w(x_\mu)}{w(x)}\Big), \quad \varphi \in \mathcal{F}, x \in X, k \in \mathbb{N},$$

then for every domain $U \subset\subset X$ such that $U \cap M \neq \varnothing$,

- *there exists a $k_0 \in \mathbb{N}$ such that $\widetilde{\varphi}_k \in \mathcal{O}(U)$ for $k \geq k_0$ and*
- *the sequence $(\widetilde{\varphi}_k)_{k=1}^\infty$ converges uniformly on U.*

Suppose for a moment that the lemma is proved. Let $\widetilde{\varphi}(x) := \lim_{k\to\infty} \widetilde{\varphi}_k(x)$, $x \in U$. Then $\widetilde{\varphi} \in \mathcal{O}(U)$. Since $x_1 = x$ for $x \in M \cap U$, we get

$$\widetilde{\varphi}(x) = \varphi(x)h(x) \prod_{\mu=2}^{\infty} \Big(1 - \frac{w(x_\mu)}{w(x)}\Big) = \varphi(x)h(x)\widetilde{w}_1(x), \quad x \in M \cap U,$$

where

$$\widetilde{w}_1(x) := \prod_{\mu=2}^{\infty} \Big(1 - \frac{w(x_\mu)}{w(x)}\Big), \quad x \in M.$$

Observe that the condition $|f_{d+1} - (\lambda_{k+1} + k^2 + 1)| < 1$ on G_k, $k \in \mathbb{N}$, implies that the function $\widetilde{w}_1$ is well-defined (cf. the estimate of the function B in the proof of Lemma 5.9). Hence $\widetilde{w}_1 \in \mathcal{O}(M)$. Notice that $\widetilde{w}_1(x) \neq 0$ for $x \in Y$.

We move to the main part of proof.
First we take $Y = Y_1 \subset M$ having a point in each connected component of M. We get a function $\widetilde{w}_1 \in \mathcal{O}(M)$ such that $\widetilde{w}_1(x) \neq 0$ for each $x \in Y_1$. In particular $M_1 := \{x \in M : \widetilde{w}_1(x) = 0\}$ is $(d-1)$-dimensional analytic subset of M. Next we take $Y_2 \subset M_1$ having a point in each connected component of $\mathrm{Reg}(M_1)$. We get $\widetilde{w}_2 \in \mathcal{O}(M)$ such that $\widetilde{w}_2(x) \neq 0$ for each $x \in Y_2$. Thus $M_2 := \{x \in M : \widetilde{w}_1(x) = \widetilde{w}_2(x) = 0\}$ is a $(d-2)$-dimensional analytic subset of M. We repeat the procedure and we obtain $\widetilde{w}_1, ..., \widetilde{w}_d \in \mathcal{O}(M)$ without common zeros on M. By Theorem 2.3 there exist $c_1, ..., c_d \in \mathcal{O}(M)$ such that $c_1\widetilde{w}_1 + ... + c_d\widetilde{w}_d = 1$ on M. Assume that h_s is constructed with respect to the family $\mathcal{F}_s := \{uc_s : u \in \mathcal{F}\}$.

We get f_s, $H_{s,k}$, $Q_{s,k}$, $\omega_{s,k}$, $\lambda_{s,k}$,$(x^k_{s,j})_{j=1}^{\lambda_{s,k}}$, $(x_{s,\nu})_{\nu=1}^\infty$, $\Xi_{s,k}(.)$, w_s, $\widetilde{w}_s$ for $s = 1, \dots d$, $k \geq 1$.

Now we are going to construct a global system of holomorphic multivalued projections on $X \longrightarrow M$ (cf. Definition 5.1). Fix arbitrary domains $U_j \subset U_{j+1} \Subset X$ such that $\bigcup_{j=1}^\infty U_j = X$, $U_j \cap M \neq \varnothing$. Let $(t_j)_{j=1}^\infty \subset \mathbb{N}$ be such that

- $f_s(U_j) \subset \mathbb{D}^d(\alpha_s(t_j))$, where $\alpha_s(t_j) \in (0, +\infty)$;
- $U_j \cap M \subset Q_{s,t_j}$;

- $t_j \le t_{j+1}, \qquad s = 1, \dots, d;$
- $t_j \to +\infty$.

Put $k_{s,j} := \lambda_{s,t_j}$. We define $\Delta_{s,j} : U_j \longrightarrow \overleftrightarrow{(M \times \mathbb{C}^n)^{k_{s,j}}}$ by

$$\Delta_{s,j}(x) := [(F_{s,1}(x), G_{s,1}(x)), \dots, (F_{s,k_{s,j}}(x), G_{s,k_{s,j}}(x))],$$

where $F_{s,m}(x) := x^m_{s,t_j}$,

$$G_{s,m}(x) := \frac{c_s(F_{s,m}(x))h_s(F_{s,m}(x))}{h_s(x)} \prod_{\mu \in \Xi^s_{t_j}(x) \setminus \{p_{j,m,s}\}} \Big(1 - \frac{w_s(x_{s,\mu})}{w_s(x)}\Big);$$

and $p_{j,m,s} \in \mathbb{N}$ is such that $x^m_{s,t_j} = x_{s,p_{j,m,s}}$.

Then $\Pi := (\Delta_{s,j})_{(s,j) \in \{1,\dots,d\} \times \mathbb{N}}$ is the global system of holomorphic multi-valued projections on $X \longrightarrow M$.

Indeed, since $U_j \subset f_s^{-1}(\mathbb{D}^d(\alpha_s(t_j)))$, then similarly as in the proof of Theorem 4.8 we show that $\Delta_{s,j}$ are holomorphic (see [**4**]). Next, we see that for $x \in M$ we have

$$\lim_{j \to \infty} \sum_{s=1}^{d} \Xi(x, \Delta_{s,j}) = \sum_{s=1}^{d} c_s(x) \widetilde{w}_s(x) = 1.$$

The construction of a global system of holomorphic projections has been finished. $\square$

Proof of the Lemma 5.9. Fix an arbitrary domain $U \subset\subset X$, $U \cap M \neq \varnothing$ and $k_0 \in \mathbb{N}$ such that $f(\overline{U}) \subset \mathbb{D}^d(\alpha(k_0))$. Let f^*_{d+1} be for a moment arbitrary and let $\varphi \in \mathcal{F}$. Take a $k \ge k_0$.

First, we are going to prove that $\widetilde{\varphi}_k \in \mathcal{O}(U)$. Note that if $x \in U$ and $j \in \Xi_k(x)$, then $x_j \in H_k \cap f^{-1}(\mathbb{D}^d(\alpha(k)) = Q_k$. Hence $\{x_j : j \in \Xi_k(x)\} = \{x^k_1, \dots, x^k_{\lambda_k}\} = \{\omega_k(f(x))\}$, $x \in U$. Moreover,

$$\widetilde{\varphi}_k(x) = w^{1-\lambda_k}(x) = \sum_{\nu=0}^{\lambda_k - 1} \overleftrightarrow{S_\nu}(\omega_k(f(x)))w^k(x), \quad x \in U,$$

where

$$S_{\lambda_k - 1}(t) := \sum_{j=1}^{\lambda_k} \varphi(t_j)h(t_j),$$

$$S_\nu(t) := (-1)^{\lambda_k - 1 - \nu} \sum_{j=1}^{\lambda_k} \varphi(t_j)h(t_j)\sigma_{k-1-\nu}(w(t_1), \dots, w(t_{j-1}), w(t_{j+1}), \dots, w(t_{\lambda_k})),$$

$$\nu = 0, \dots, \lambda_k - 2,\ t = (t_1, \dots t_{\lambda_k}) \in Q_k^{\lambda_k},$$

and $\sigma_1, \dots, \sigma_{\lambda_k-1} : \mathbb{C}^{\lambda_k-1} \longrightarrow \mathbb{C}$ are standard symmetric polynomials. Consequently, by Proposition 4.10 we conclude that $\widetilde{\varphi}_k \in \mathcal{O}(U)$.

Now we are going to find a function $f^*_{d+1} \in \mathcal{O}(U)$ (independent of U) such that $(\widetilde{\varphi}_k)_{k=1}^{\infty}$ converges uniformly on U.

We construct f^*_{d+1} via Theorem 2.6 in such a way that $|f^*_{d+1} - k^2\beta_k\lambda_k - 1| < 1$ on G_k, where $\beta_k \geq \sup\{\sup_{G_k}|\varphi| : \varphi \in \mathcal{F}\}$. Our aim is to prove that $\widetilde{\varphi}_l(x) - \widetilde{\varphi}_k(x) \longrightarrow 0$ uniformly on U when $l > k \longrightarrow +\infty$. Take $l > k \geq k_0$. For $x \in U$ write

$$\begin{aligned}\widetilde{\varphi}_l(x) - \widetilde{\varphi}_k(x) = &\sum_{j \in \Xi_l(x) \setminus \Xi_k(x)} \varphi(x_j)h(x_j) \prod_{\substack{\mu \in \Xi_l(x)\\ \mu \neq j}} \Big(1 - \frac{w(x_\mu)}{w(x)}\Big)\\ &+ \sum_{j \in \Xi_k(x)} \varphi(x_j)h(x_j) \prod_{\substack{\mu \in \Xi_k(x)\\ \mu \neq j}} \Big(1 - \frac{w(x_\mu)}{w(x)}\Big)\\ &\cdot \Big(\Big(\prod_{\substack{\mu \in \Xi_l(x) \setminus \Xi_k(x)\\ \mu \neq j}} \Big(1 - \frac{w(x_\mu)}{w(x)}\Big)\Big) - 1\Big) = I_{k,l}(x) + J_{k,l}(x).\end{aligned}$$

We have

$$\begin{aligned}|I_{k,l}(x)| &\leq \Big(\sum_{j \notin \Xi_k(x)} |\varphi(x_j)h(x_j)|\Big) \cdot \prod_{\mu \in \mathbb{N}} \Big(1 + \frac{|w(x_\mu)|}{|w(x)|}\Big) =: A_k(x)B(x),\\ |J_{k,l}(x)| &\leq \Big(\sum_{j \in \mathbb{N}} |\varphi(x_j)h(x_j)|\Big) \cdot \Big(\prod_{\mu \in \mathbb{N}} \Big(1 + \frac{|w(x_\mu)|}{|w(x)|}\Big)\Big)\\ &\quad \cdot \Big(\Big(\prod_{\mu \notin \Xi_k(x)} \Big(1 + \frac{|w(x_\mu)|}{|w(x)|}\Big)\Big) - 1\Big) =: C(x)D(x)(E_k(x) - 1).\end{aligned}$$

Observe that $M \cap f^{-1}(\mathbb{D}^d(\alpha(k_0))) \subset Q_{k_0} \cup \bigcup_{s=k_0}^{\infty} \mathrm{int} G_s$. Let

$$\gamma := \max_{\overline{U}} \mathrm{Re} f_{d+1}, \quad \delta := \max_{H_{k_0}}(-\mathrm{Re} f_{d+1}).$$

Observe that if $x \in U$ and $x_\mu \in \mathrm{int} G_s$, then we have

$$\log\Big(1 + \frac{|w(x_\mu)|}{|w(x)|}\Big) \leq \frac{|w(x_\mu)|}{|w(x)|} = \mathrm{e}^{-\mathrm{Re} f_{d+1}(x_\mu) + \mathrm{Re} f_{d+1}(x)} \leq \mathrm{e}^{-\lambda_{s+1} - s^2 + \gamma} \leq \frac{\mathrm{e}^{\gamma}}{\lambda_{s+1}s^2}.$$

If $x_\mu \in Q_{k_0}$, then

$$\log\Big(1 + \frac{|w(x_\mu)|}{|w(x)|}\Big) \leq \frac{|w(x_\mu)|}{|w(x)|} = \mathrm{e}^{-\mathrm{Re} f_{d+1}(x_\mu) + \mathrm{Re} f_{d+1}(x)} \leq \mathrm{e}^{\delta + \gamma}.$$

Thus for all $x \in U$ we have

$$\begin{aligned}\log B(x) &= \sum_{\mu\colon x_\mu \in Q_{k_0}} \log\left(1+\frac{|w(x_\mu)|}{|w(x)|}\right) + \sum_{s=k_0}^{\infty} \sum_{\mu\colon x_\mu \in \mathrm{int} G_s} \log\left(1+\frac{|w(x_\mu)|}{|w(x)|}\right)\\ &\le \lambda_{k_0} \mathrm{e}^{\delta+\gamma} + \sum_{s=k_0}^{\infty} \frac{\mathrm{e}^{\gamma}}{s^2},\end{aligned}$$

and therefore the function B is uniformly bounded on U.

Similarly,

$$\begin{aligned}C(x) &= \sum_{\mu\colon x_\mu \in Q_{k_0}} |\varphi(x_\mu)h(x_\mu)| + \sum_{s=k_0}^{\infty} \sum_{\mu\colon x_\mu \in \mathrm{int} G_s} |\varphi(x_\mu)h(x_\mu)|\\ &\le M + \sum_{s=k_0}^{\infty} \sum_{\mu\colon x_\mu \in \mathrm{int} G_s} \beta_s \mathrm{e}^{-\mathrm{Re} f^*_{d+1}(x_\mu)} \le M + \sum_{s=k_0}^{\infty} \sum_{\mu\colon x_\mu \in \mathrm{int} G_s} \beta_s \mathrm{e}^{-s^s \beta_s \lambda_s}\\ &\le M + \sum_{s=k_0}^{\infty} \sum_{\mu\colon x_\mu \in \mathrm{int} G_s} \beta_s \frac{1}{\beta_s \lambda_s s^2} \le M + \sum_{s=k_0}^{\infty} \frac{1}{s^2},\end{aligned}$$

where $M := \lambda_{k_0} \sup_{Q_{k_0}} |\varphi h|$. On the other hand,

$$\begin{aligned}A_k(x) &= \sum_{s=k}^{\infty} \sum_{j\colon x_j \in \mathrm{int} G_s} \beta_s |h(x_j)| = \sum_{s=k}^{\infty} \sum_{j\colon x_j \in \mathrm{int} G_s} \beta_s \mathrm{e}^{-\mathrm{Re} f^*_{d+1}(x_j)}\\ &\le \sum_{s=k}^{\infty} \sum_{j\colon x_j \in \mathrm{int} G_s} \beta_s \mathrm{e}^{-\beta_s \lambda_{s+1} - s^2} \le \sum_{s=k}^{\infty} \sum_{j\colon x_j \in \mathrm{int} G_s} \beta_s \frac{1}{\beta_s \lambda_{s+1} s^2} \le \sum_{s=k}^{\infty} \frac{1}{s^2}.\end{aligned}$$

which proves that $A_k(x)B(x) \longrightarrow 0$ uniformly on U, when $k \longrightarrow +\infty$. We have proved that the product $\prod_{\mu\in\mathbb{N}}\left(1+\frac{|w(x_\mu)|}{|w(x)||}\right)$ converges uniformly on U. In particular,

$$E_k(x) = \frac{\prod_{\mu\in\mathbb{N}}\left(1+\frac{|w(x_\mu|}{|w(x)|}\right)}{\prod_{\mu\in\Xi_k(x)}\left(1+\frac{|w(x_\mu)|}{|w(x)|}\right)} \longrightarrow 1, \quad \text{uniformly for } x \in U.$$

□

Observe that by Remark 5.4, we have the extension operator $L_\Pi : \mathcal{F} \longrightarrow \mathcal{O}(X)$. Now we are going to check its continuity. Note that $\mathcal{F}$ and $\mathcal{O}(X)$ are endowed with the locally uniform convergence topologies.

Continuity of L_Π. Let $\mathcal{F} \ni \varphi_t \longrightarrow \varphi \in \mathcal{F}$ locally uniformly. Fix $\varepsilon > 0$ and compact set $K \subset X$. Observe that $\mathcal{F}' := \mathcal{F} \cup (\varphi_t - \varphi)_{t=1}^{\infty}$ is also locally bounded. Moreover, there is $L'_\Pi : \mathcal{F}' \longrightarrow \mathcal{O}(X)$ extension operator such that $L_\Pi = L'_\Pi$ on $\mathcal{F}$. Indeed, the map f^*_{d+1} is good for the both families $\mathcal{F}$ and $\mathcal{F}'$ if we take $\beta_k = \beta'_k \geq 2\sup\{\sup_{G_k} |\varphi| : \varphi \in \mathcal{F}\}$, where β'_k is constant in the construction of the operator L'_Π. For $x \in K$ we have

$$\begin{aligned}
|L_\Pi(\varphi_t) - L_\Pi(\varphi)|(x) &= |L'_\Pi(\varphi_t - \varphi)|(x) \\
&= \Big| \sum_{s=1}^{d} \frac{1}{h_s(x)} \sum_{j=1}^{\infty} (\varphi_t - \varphi)(x_{s,j}) c_s(x_{s,j}) h_s(x_{s,j}) \prod_{\substack{\mu\in\mathbb{N}\\ \mu\neq j}} \Big(1 - \frac{w_s(x_{s,\mu})}{w_s(x)}\Big)\Big| \\
&\leq \sum_{s=1}^{d} \frac{1}{h_s(x)} \prod_{\mu\in\mathbb{N}} \Big(1 + \frac{|w_s(x_{s,\mu})|}{|w_s(x)|}\Big) \sum_{j=1}^{\infty} |(\varphi_t - \varphi)(x_{s,j}) c_s(x_{s,j}) h_s(x_{s,j})|.
\end{aligned}$$

Let $f(K) \subset \mathbb{D}^d(\alpha(k_0))$ and $k_1 \geq k_0$, where $k_0, k_1 \in \mathbb{N}$. By the proof of the previous lemma we get the following estimate

$$\prod_{\mu\in\mathbb{N}} \Big(1 + \frac{|w_s(x_{s,\mu})|}{|w_s(x)|}\Big) \leq \lambda_{k_0} \mathrm{e}^{\delta+\gamma} + \sum_{s=k_0}^{\infty} \frac{\mathrm{e}^{\gamma}}{s^2}.$$

Since K is compact, we conclude that the map $x \longmapsto \frac{1}{h_s(x)} \prod_{\mu\in\mathbb{N}} \Big(1 + \frac{|w_s(x_{s,\mu})|}{|w_s(x)|}\Big)$ is bounded on K. On the other hand,

$$\sum_{j=1}^{\infty} |(\varphi_t - \varphi)(x_{s,j}) c_s(x_{s,j}) h_s(x_{s,j})| \leq M + \sum_{s=k_1}^{\infty} \frac{1}{s^2},$$

where $M := \lambda_{k_1} \sup_{Q_{k_1}} |(\varphi_t - \varphi) c_s h_s|$. Now we observe that if k_1 and t are sufficiently large, we obtain

$$\|L_\Pi(\varphi_t) - L_\Pi(\varphi)\|_K \leq \varepsilon.$$

□

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References

1. Arens R., *Dense inverse limit rings*, Michigan Math. J., **5** (1958), 169–182.
2. Bishop E., *Some global problems in the theory of functions of several complex variables*, Amer. J. Math., **83** (1961), 479–498.
3. Chirka E.M, *Complex analytic sets*, Kluwer Academic Publishers 1989.

4. Drzyzga K., *A remark on Bishop's multivalued projections*, Univ. Iag. Acta Math., **53** (2016), 13–20.
5. Gunning R., Rossi H., *Analytic Functions of Several Complex Variables*, Prentice-Hall, Engelewood Cliffs, 1965.
6. Łojasiewicz S., *Introduction to Complex Analytic Geometry*, Birkhäuser, 1991.
7. Whitney H., *Complex Analytic Varieties*, Addison-Wesley Publishing Company, 1972.

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Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Mathematics
Łojasiewicza 6
30-348 Kraków, Poland
e-mail: kamil.drzyzga@gmail.com